GODDARD
GRANT
IN-32-CR
12240
P.71

CONCATENATED CODED MODULATION TECHNIQUES

Technical Report
to
NASA
Goddard Space Flight Center
Greenbelt, Maryland 20771

Grant Number NAG 5-931

Report Number NASA 91-001

Shu Lin
Principal Investigator
Department of Electrical Engineering
University of Hawaii at Manoa
Honolulu, Hawaii 96822

April 20, 1991

188216

CONCATENATED CODED MODULATION TECHNIQUES

CONTENTS

1. Introduction

CODED MODUALTION ALONE

- To achieve a 3 to 5 dB coding gain and moderate reliability, the decoding complexity is quite modest.
- In fact, to achieve a 3 dB coding gain, the decoding complexity is quite simple, no matter whether **trellis coded modulation** (**TCM**) or **block coded modulation** (**BCM**) is used.
- However, to achieve coding gains exceeding 5dB, the decoding complexity increases **drastically** , and the implementation of the decoder becomes very expensive and unpractical.

A BASIC QUESTION

- How can we achieve large coding gains and high reliability by using coded modulation with reduced decoding complexity ?

AN ANSWER

- Use coded modulation **in conjunction** with **concatenated** (or **cascaded**) coding.

- A good short **bandwidth efficient** modulation code (**trellis** or **block**) is used as the inner code and a relatively powerful Reed-Solomon (RS) code is used as the outer code.

- With properly chosen inner and outer codes, a concatenated coded modulation scheme not only achieve large coding gains and high reliability with good bandwidth efficiency but can also be practically implemented.

- This **combination** of coded modulation and concatenation coding really offers a way of achieving the **best** of four worlds, reliability,coding gain,bandwidth efficiency and decoding complexity.

2. Single-Level Concatenated Coded Modulation

THE OVERALL CONCATENATED CODED MODULATION SCHEME

- The outer code C_2 is an (n_2, k_2) RS code over GF(2^b), which is designed to correct t_2 or fewer symbol errors with $0 \leq t_2 \leq \lfloor (n_2 - k_2)/2 \rfloor$.
- The inner code C_1 is a bandwidth efficient modulation code of length n_1 and dimension $k_1 = mb$.
- The outer code C_2 is interleaved to a depth of m as shown in Figure 1.
- The encoding consists of two stages, the outer and inner encodings as shown in Figure 2.
- The decoding consists of two stages, the inner and outer decodings.

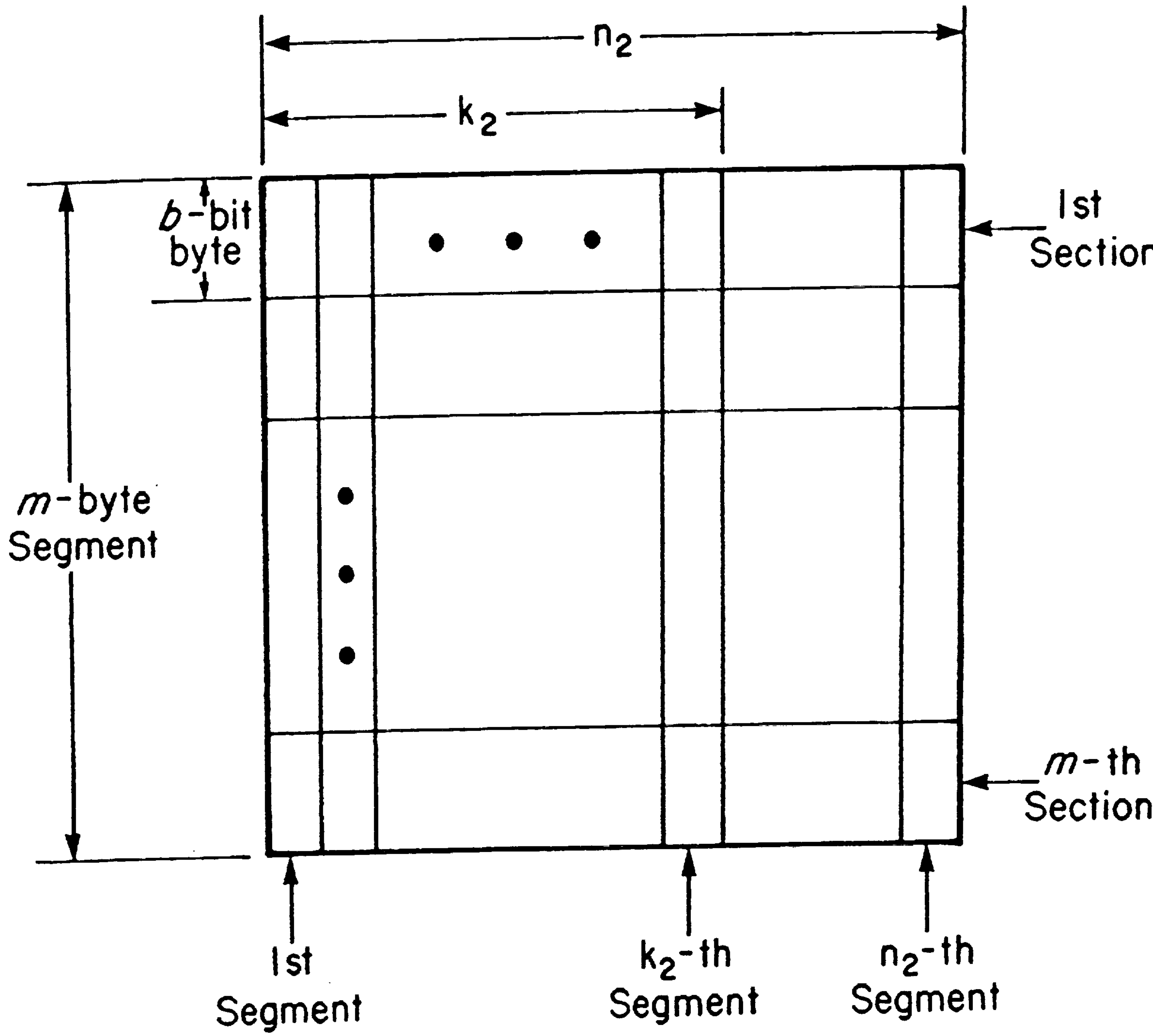


Figure 1 A Segment-Array

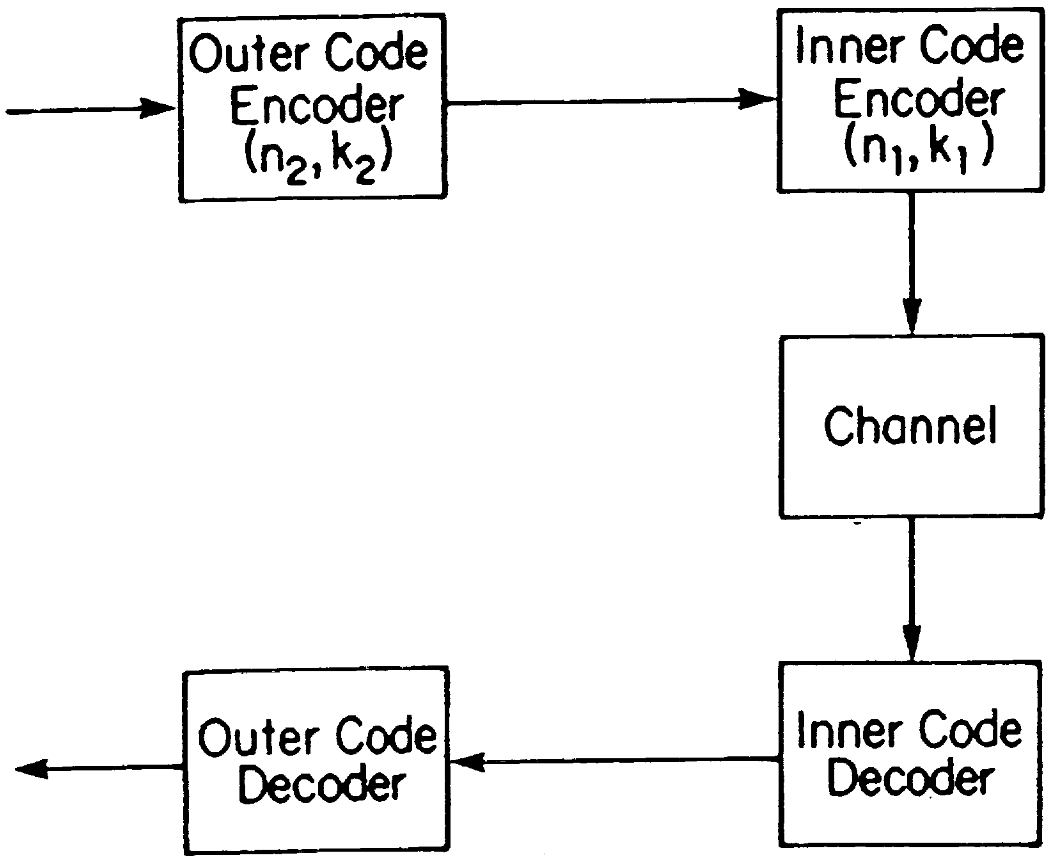


Figure 2 A cascaded coding system

A Concatenated Coded Modulation System

- For NASA high-speed satellite communications for large data file transfer where very high reliability is required.
- The outer code C_2 is the NASA standard (255,223) RS code over GF(2^8) which has minimum Hamming distance 33. It is used to correct up to 16 symbol errors.
- The inner code is an 8-PSK modulation code with $n_1 = 8$, $k_1 = 16, D[C_1] = 4, R[C_1] = 1$ and $\gamma[C_1] = 3$ dB (over uncoded QPSK).
- The outer code is interleaved to a depth of m = 2.
- The overall effective rate of the scheme is

$$R_{eff} = (k_2/n_2) \cdot R[C_1] = 0.875.$$

- The inner code has a 4-state trellis structure and can be decoded with a soft-decision Viterbi decoder.
- Error performance is shown in Figures 3 - 7.

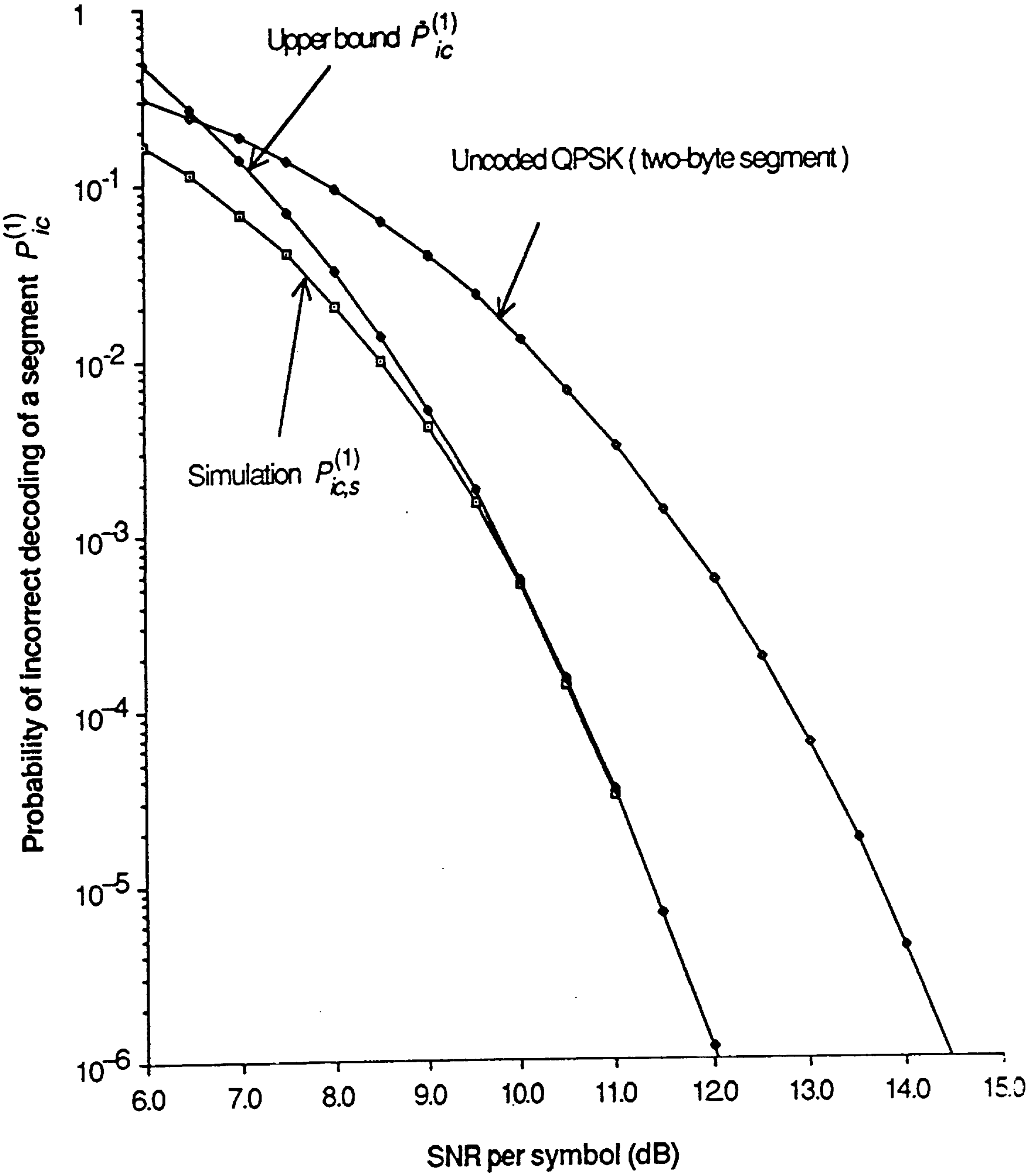

Upper bound $\hat{P}_{ic}^{(1)}$
Uncoded QPSK (two-byte segment)
Simulation $P_{ic,s}^{(1)}$
Probability of incorrect decoding of a segment $P_{ic}^{(1)}$
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
6.0
7.0
8.0
9.0
10.0
11.0
12.0
13.0
14.0
15.0
SNR per symbol (dB)

Figure 3

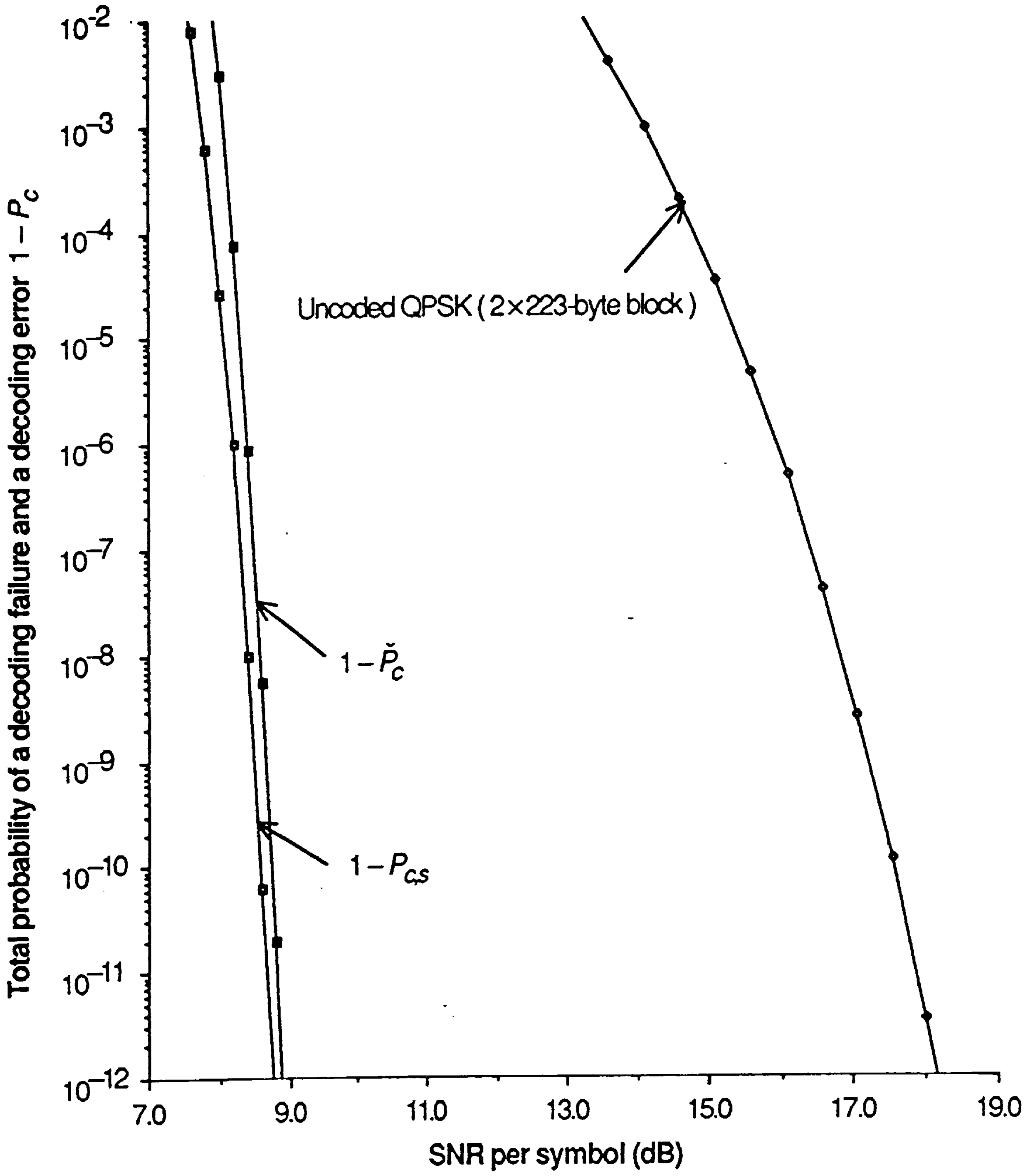
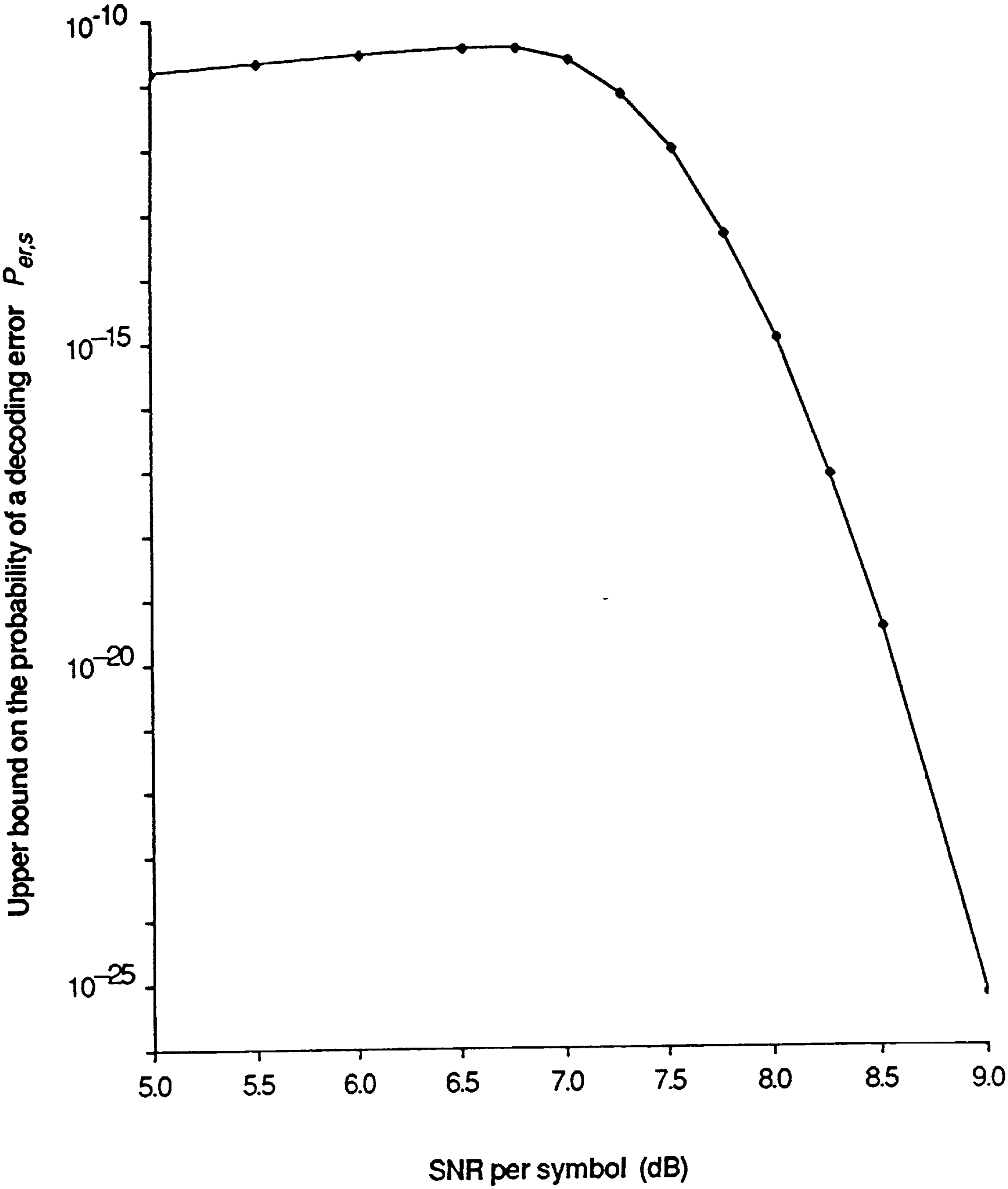
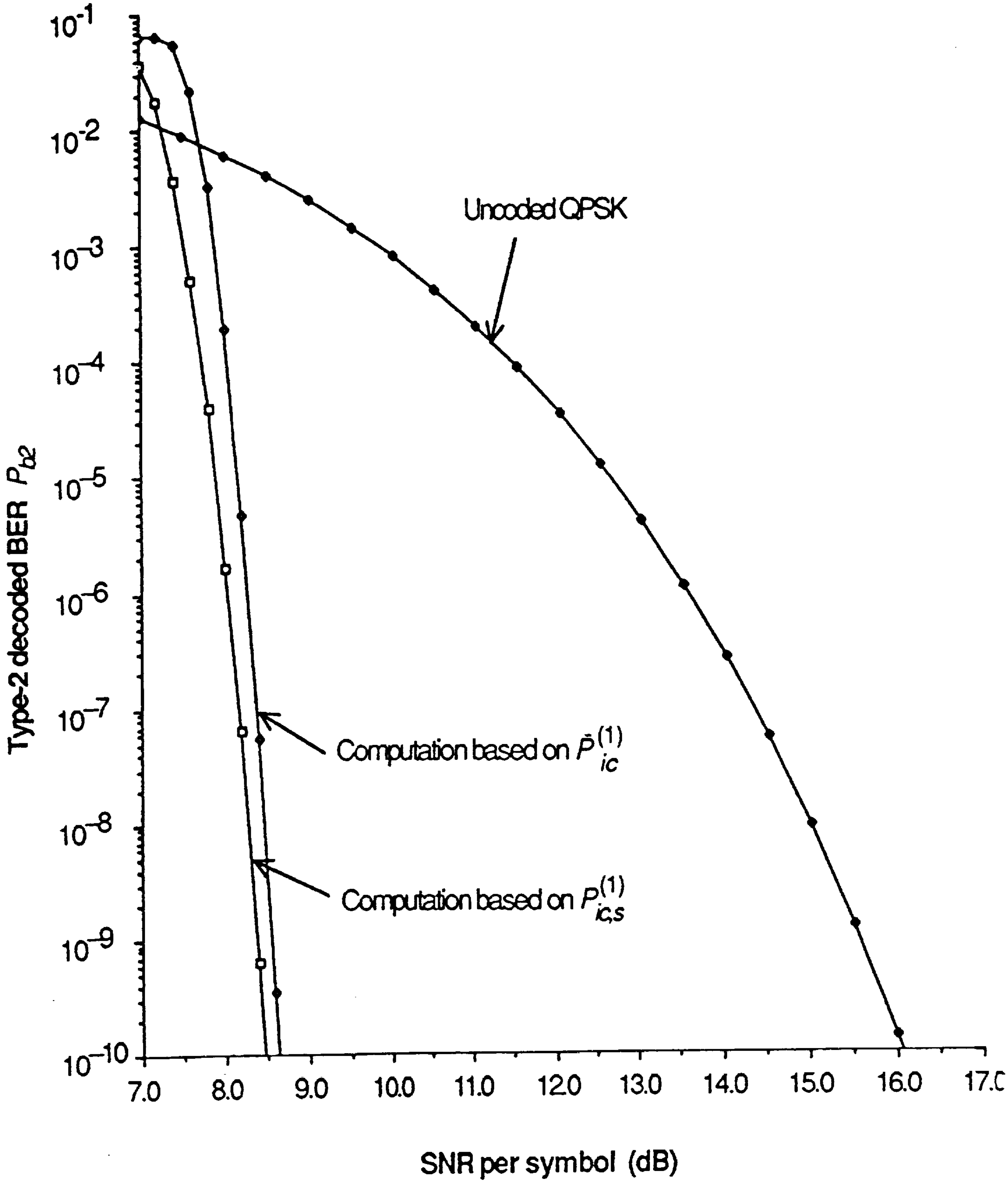

Total probability of a decoding failure and a decoding error $1-P_c$
SNR per symbol (dB)
Uncoded QPSK (2×223-byte block)
$1-\check{P}_c$
$1-P_{c,s}$
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{-12}
7.0
9.0
11.0
13.0
15.0
17.0
19.0

Figure 4



Upper bound on the probability of a decoding error $\hat{P}_{er,s}$
10^{-10}
10^{-15}
10^{-20}
10^{-25}
5.0
5.5
6.0
6.5
7.0
7.5
8.0
8.5
9.0
SNR per symbol (dB)

Figure 5



Type-2 decoded BER P_{b2}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
Uncoded QPSK
Computation based on $\bar{P}_{ic}^{(1)}$
Computation based on $P_{ic,s}^{(1)}$
7.0
8.0
9.0
10.0
11.0
12.0
13.0
14.0
15.0
16.0
17.0
SNR per symbol (dB)

Figure 6

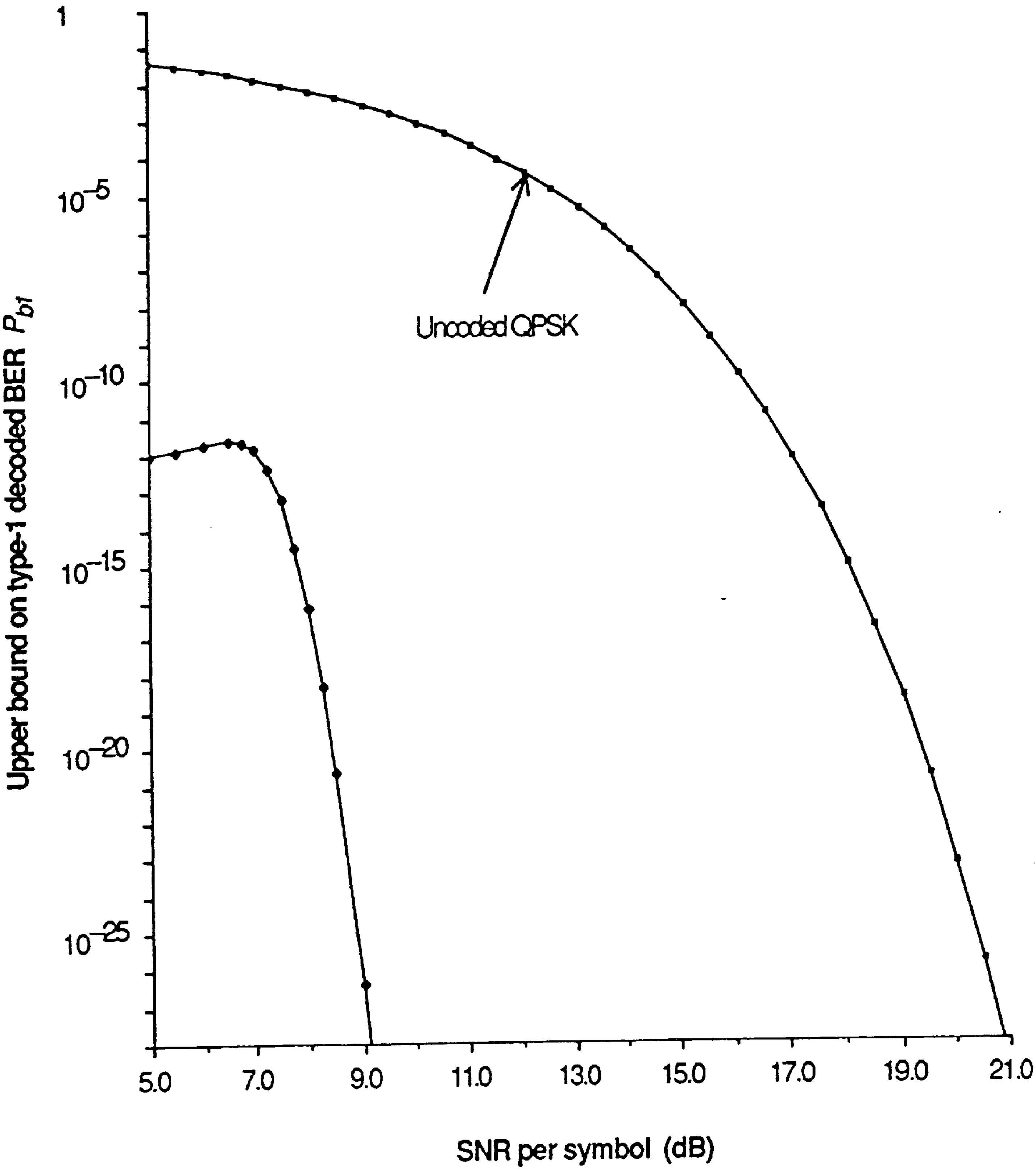

1
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
Upper bound on type-1 decoded BER P_{b1}
Uncoded QPSK
5.0
7.0
9.0
11.0
13.0
15.0
17.0
19.0
21.0
SNR per symbol (dB)

Figure 7

ERROR PERFORMANCE

- With SNR = 9 dB/symbol (6.57 dB/infor. bit),

$$P_{er} \leq 6.28 \times 10^{-25}$$

$$1 - P_c \leq 4.95 \times 10^{-16}$$

- With SNR = 10 dB /symbol (5.57 dB / infor. bit) ,

$$P_{er} \leq 6.80 \times 10^{-41}$$

and 1 - P_c is very small.

CODING GAIN OVER QPSK

- At the block-error rate $= 10^{-7}$,

$$G_B = 8 \text{ dB/symbol.}$$

- At the block-error rate $= 10^{-10}$,

$$G_B = 9 \text{ dB/symbol.}$$

- At the bit-error rate $P_{b1} = 10^{-12}$,

$$G_{b1} = \ 9.80 \text{ dB/symbol (9.20 dB/infor. bit).}$$

The required SNR to achieve $P_{b1} = 10^{-12}$ is 7.10 dB /symbol (4.60 dB /infor. bit).

- At the bit-error rate $P_{b2} = 10^{-6}$,

$$G_{b2} = 5.52 \text{ dB /symbol (4.94 dB/infor. bit)}.$$

The required SNR to achieve $P_{b2} = 10^{-6}$ is 8.04 dB /symbol (5.61 dB /infor. bit).

- At the bit-error rate $P_{b2} = 10^{-10}$,

$$G_{b2} = 7.60 \text{ dB/symbol (7.02 dB/infor. bit)}.$$

The required SNR to achieve $P_{b2} = 10^{-10}$ is 8.50 dB / symbol (6.07 dB / infor. bit).

3. Multi-Level Concatenated Coded Modulation

- Coded modulation in conjunction with concatenation is a powerful technique for achieving large coding gain (or high reliability) with high spectral efficiency and reduced decoding complexity.
- If concatenation is carried out in multiple levels, further improvement in spectral efficiency can be obtained.
- In this presentation, we describe a multi-level concatenated coded modulation scheme. In this scheme, Reed - Solomon (or Maximum - Distance - Separable) codes are concatenated with coset codes derived from a linear bandwidth efficient modulation code and its linear proper subcodes in multiple levels.
- The proposed scheme can be used to construct multi-level multi-dimensional TCM codes.

Outer Codes

- For $1 \leq i \leq l$, let Λ_i be an (N_i, K_i) Reed - Solomon (or shortened Reed - Solomon) code over $GF(2^{m_i})$ with minimum Hamming distance D_i , where $D_i = N_i - K_i + 1$. These l Reed - Solomon codes will be used as outer codes in l levels of concatenation .

Base Inner Code

- Let C_1 be a linear block modulation code over a certain signal set S with length n_1 , dimension k_1 and **minimum squared Euclidean distance** δ_1 , where

$$k_1 = m_1 + m_2 + \cdots m_l \tag{1}$$

- For $1 \leq i \leq l$, let C_i be a linear **proper subcode** of C_{i-1} with dimension

$$k_i = k_{i-1} - m_{i-1} \tag{2}$$

and minimum squared Euclidean distance δ_i .

- From (1) and (2) , we see that

$$k_2 = m_2 + m_3 + \cdots + m_{l-1} + m_l$$

$$k_3 = m_3 + m_4 + \cdots + m_l$$

$$\vdots$$

$$k_l = m_l$$

- We also note that $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_l$.

- The coset codes formed from C_1 and its subcodes C_2, C_3, $\cdots$ C_l will be used as the inner codes in the proposed multi-level concatenation coding scheme .

Code partition and Coset Codes

First Partition

- Partition C_1 into 2^{m_1} cosets modulo C_2.
- Let C_1 / C_2 denote the set of cosets of C_1 modulo C_2 .
- The minimum squared Euclidean distance of each coset in C_1 / C_2 is δ_2.
- The minimum (squared) separation between two cosets in C_1 / C_2 is δ_1.
- C_1 / C_2 is called the **coset code** of C_1 modulo C_2 .

Second Partition

- Partition each coset in $C_1 \,/\, C_2$ into 2^{m_2} cosets modulo C_3.
- Let $C_1 \,/\, C_2 \,/\, C_3$ denote the set of cosets of a coset in $C_1 \,/\, C_2$ modulo C_3.
- The minimum squared Euclidean distance of a coset in $C_1 \,/\, C_2 \,/\, C_3$ is δ_3.
- The minimum (squared) separation among the cosets of a coset in $C_1 \,/\, C_2$ modulo C_3 is δ_2.
- $C_1 \,/\, C_2 \,/\, C_3$ is called the **coset code** of $C_1 \,/\, C_2$ modulo C_3.

The i-th Partition

- For $1 \leq i \leq l$, let $C_1 \;/\; C_2 \;/ \cdots /\; C_i$ be the **coset code** of $C_1 \;/\; C_2 \;/ \cdots /\; C_{i-1}$ modulo C_i.
- Partition each coset in $C_1 \;/\; C_2 \;/ \cdots /\; C_i$ into 2^{m_i} cosets modulo C_{i+1}.
- Let $C_1 \;/\; C_2 \;/ \cdots /\; C_{i+1}$ denote the set of cosets of a coset in $C_1 \;/\; C_2 \;/ \cdots /\; C_i$ modulo C_{i+1}.
- The minimum (squared) Euclidean distance of a coset in $C_1 \;/\; C_2 \;/ \cdots /\; C_{i+1}$ is δ_{i+1}.
- The minimum squared separation among the cosets of a coset in $C_1 \;/\; C_2 \;/ \cdots /\; C_i$ modulo C_{i+1} is δ_i.

The l-th Partition

- Each coset in $C_1 \ / \ C_2 \ / \cdots / \ C_l$ consists of 2^{m_l} codewords in C_1.
- Partition each coset in $C_1 \ / \ C_2 \ / \cdots / \ C_l$ into 2^{m_l} cosets modulo $C_{l+1} = \{\bar{0}\}$.
- Each coset in $C_1 \ / \ C_2 \ / \cdots / \ C_{l+1}$ consists of only one codeword in C_1. The minimum squared Euclidea distane of each coset is $\delta_{l+1} = \infty$.
- The minimum separation among the cosets of a coset in $C_1 \ / \ C_2 \ / \cdots / \ C_l$ modulo C_{l+1} is δ_l.

Remark

- The partition chain results in a sequence of coset codes , C_1/C_2 , $C_1/C_2/C_3$, $\cdots$, $C_1 \ / \ C_2 \ / \cdots / \ C_{l+1}$.
- These l coset codes are used as inner codes in the proposed multi-level concatenation scheme.

Encoding

- An organization of the overall encoder for a l - level concatenated code modulation system is shown in Figure 8.
- Every inner code encoder, except the first-level, has two inputs, one from the output of an outer code encoder and one from the output of the inner code encoder of the **preceding** level.

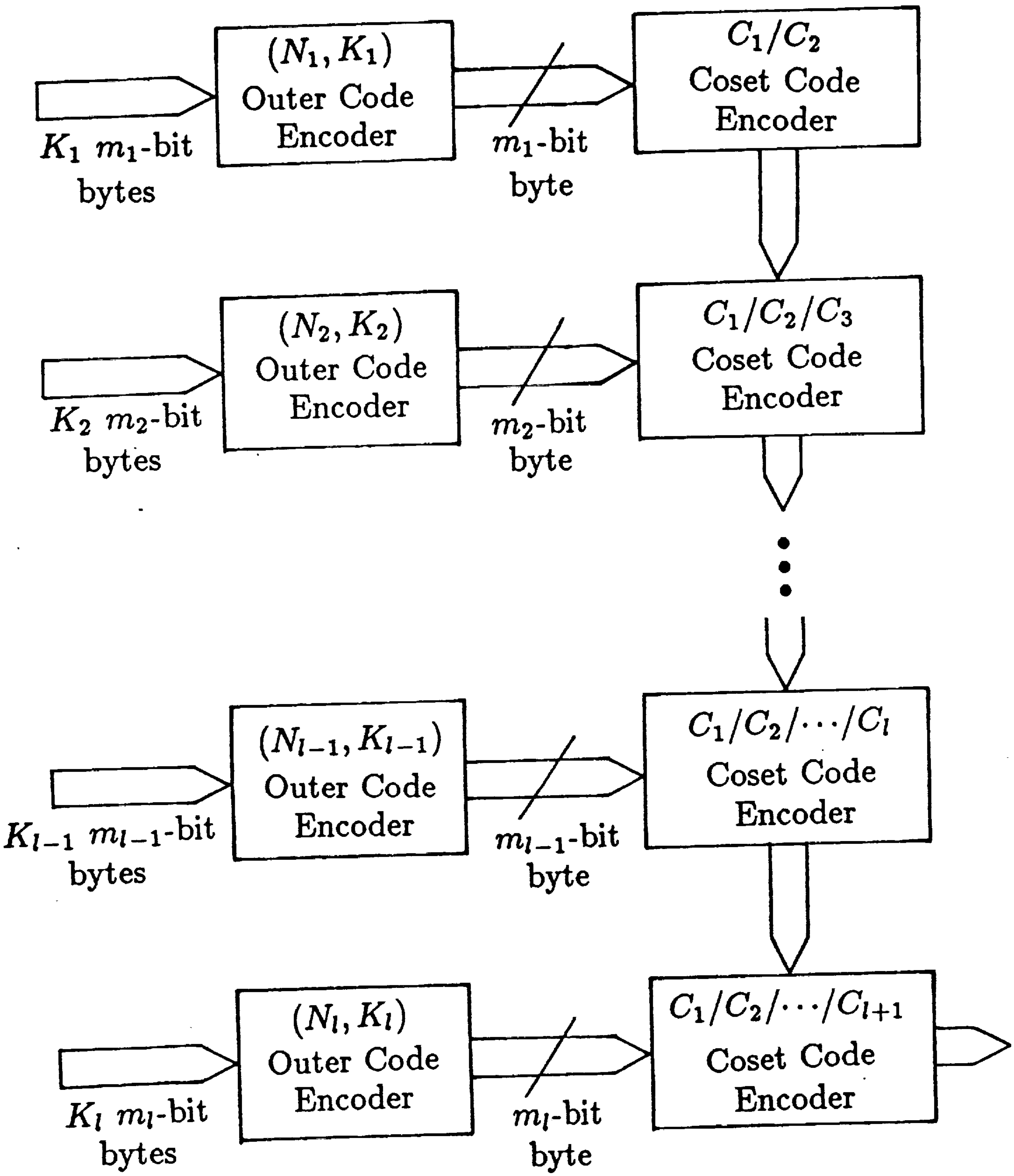


Figure 8 An overall multi-level concatenation encoder

- For $1 \leq i \leq l$, the i-th level encoding is accomplished in two steps :

  (1) Outer code encoding - A message of K_i m_i-bit bytes is encoded into a codeword of N_i m_i-bit bytes in the i-th level outer code Λ_i.

  (2) Inner code encoding - The **input coset** from the $(i-1)$-th level inner encoder is partitioned into 2^{m_i} cosets modulo C_{i+1}. Each m_i-bit byte input from the i-th level outer code encoder is encoded into a coset in the coset code $C_1/C_2/\cdots/C_{i+1}$. Therefore the output of the i -th level inner code encoder is a sequence of cosets from $C_1/C_2/\cdots/C_{i+1}$.

- The output of the l -th level inner code encoder is a sequence of codewords from base inner code C_1 .

A Special Case

- For the purpose of implementation, we choose all the outer codes of the same length, say N.
- For this special case, the overall concatenated code $\tilde{C}$ is a modulation code over the signal set S with length nN, dimension

$$\tilde{K} = m_1 K_1 + m_2 K_2 + \cdots m_l K_l,$$

and minimum squared Euclidean distance

$$\tilde{\delta} = \min_{1 \leq i \leq l} \{D_i \delta_i\}$$

- The spectral efficiency and effective rate of $\tilde{C}$ are :

$$\eta[\tilde{C}] \triangleq \frac{\tilde{K}}{nN} \text{ bits / signal}$$

and

$$R[\tilde{C}] \triangleq \frac{\tilde{K}}{2nN} \text{ bits / dimension}$$

Example I

- Suppose we want to design a two-level concatenated coded modulation system with $m_1 = m_2 = 8$.

- We choose the following two codes as the outer codes :

  (1) The first -level outer code Λ_1 is the $(255, 223)$ RS code over $GF(2^8)$. This code is the **NASA standard** with Hamming distance $D_1 = 33$.

  (2) The second-level outer code Λ_2 is the $(255, 239)$ RS code over $GF(2^8)$ with minimum Hamming distance $D_2 = 17$.

- The basic 3 -level 8 - PSK modulation code ,

$$C_1 = P_8^{\perp} * P_8 * V_8$$

  is chosen as the base inner code where (1) P_8 consists of all the binary 8 -tuples of even weight ; (2) $P_8^{\perp}$ is the dual code of P_8 (consisting of only the **all-zero** and **all-one** 8-tuples); and V_8 is the vector space of all 8−tuples over $GF(2)$. Each codeword in C_1 consists of eight 8− PSK signals which carry 16 information bits. The minimum squared Eiclidean distance of C_1 is 4. The code has a 4 -state 8-section trellis diagram.

- Let C_2 be the following basic 3-level 8-PSK modualation code ,

$$C_2 = \{\bar{0}\} * P_8^{\perp} * P_8.$$

  Then C_2 is a linear proper subcode of C_1. C_2 has dimension 8 and minimum squared Euclidean distance 8.

- The coset code C_1/C_2 consists 2^8 cosets modulo C_2.

- The coset code $C_1/C_2/C_3$ consists of 2^8 codewords from C_1 , where $C_3 = \{\bar{0}\}$.

- The overall encoder is shown in Figure 9.
- The overall modulation code $\tilde{C}$ has length 2040 , dimension 3696 , and minimum squared Euclidean distance $\tilde{\delta} = 132$.
- The spectral efficiency and effective rate of $\tilde{C}$ are :

$$\eta[\tilde{C}] = \frac{3696}{2040} = 1.818 \text{ bits / signal}$$

and

$$R[\tilde{C}] = 0.906 \text{ bits / dimension}$$

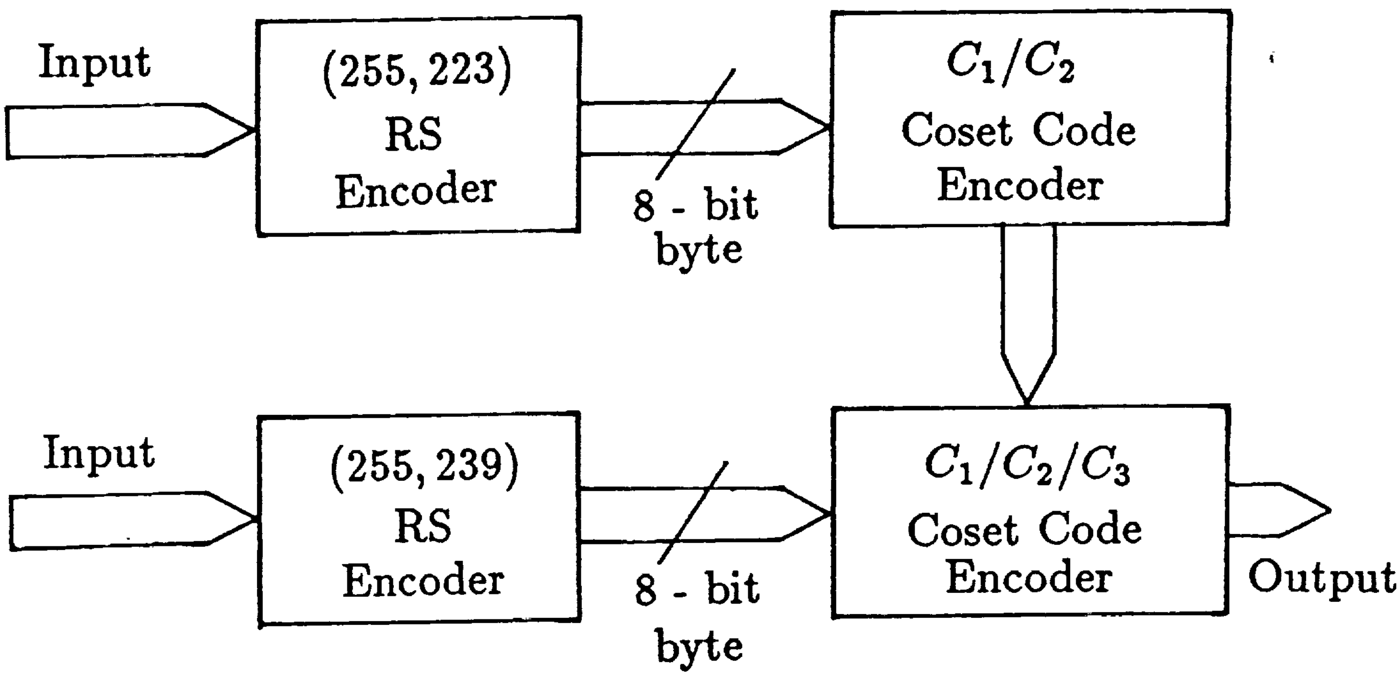


Figure 9 An encoder for the 2-level concatenated coded modulation system given in Example I

Example II

Consider the design of a 4-level concatenated code modulation system with $m_1 = m_2 = m_3 = m_4 = 8$.

- Outer codes are the following RS codes over $GF(2^8)$:

$$\Lambda_1 = (255, 223) \text{ RS code with } D_1 = 33$$

$$\Lambda_2 = (255, 239) \text{ RS code with } D_2 = 17$$

$$\Lambda_3 = (255, 245) \text{ RS code with } D_3 = 11$$

$$\Lambda_4 = (255, 247) \text{ RS code with } D_4 = 9$$

- The base inner code and its subcodes are the following basic 3-level 8-PSK modulation codes of length 15 :

$$C_1 = s - RM_{4,1} * P_7 \circ P_8 \ * V_{15}$$

$$C_2 = \{\bar{0}\} * s - RM_{4,2} * P_{15}$$

$$C_3 = \{\bar{0}\} * BCH_{15,6,6} * s - RM_{4,2}$$

$$C_4 = \{\bar{0}\} * s - RM_{4,1} * s - RM_{4,1}$$

where (1) $s{-}RM_{m,r}$ denotes a shortened Reed - Muller code of length $2^m - 1$ and minimum hamming distance 2^{m-r}, (2) $P_7 \circ P_8$ denotes the concatenation of P_7 and P_8, and (3) $BCH_{n,k,d}$ denotes a code equivalent to the (n,k) primitive BCH code or its even-weight subcode with designed minimum Hamming distance d.

- The dimensions of C_1, C_2, C_3 and C_4 are : $k_1 = 32, k_2 = 24, k_3 = 16$ and $k_4 = 8$. Their minimum squared Euclidean distances are : $\delta_1 = 4, \delta_2 = 8, \delta_3 = 12$ and $\delta_4 = 16$.
- The overall code $\tilde{C}$ is a 8-PSK modulation code with length 3825 , dimension 7632, and minimum squared Euclidean distance 132.
- The spectral efficiency and effective rate of $\tilde{C}$ are :

$$\eta[\tilde{C}] = 1.995 \text{ bits / signal}$$

and

$$R[\tilde{C}] = 0.998 \text{ bits / dimension}$$

- The overall encoder is shown in Figure 10.
- This system provides large coding gain over the uncoded QPSK system with only 0.2 % bandwidth expansion.

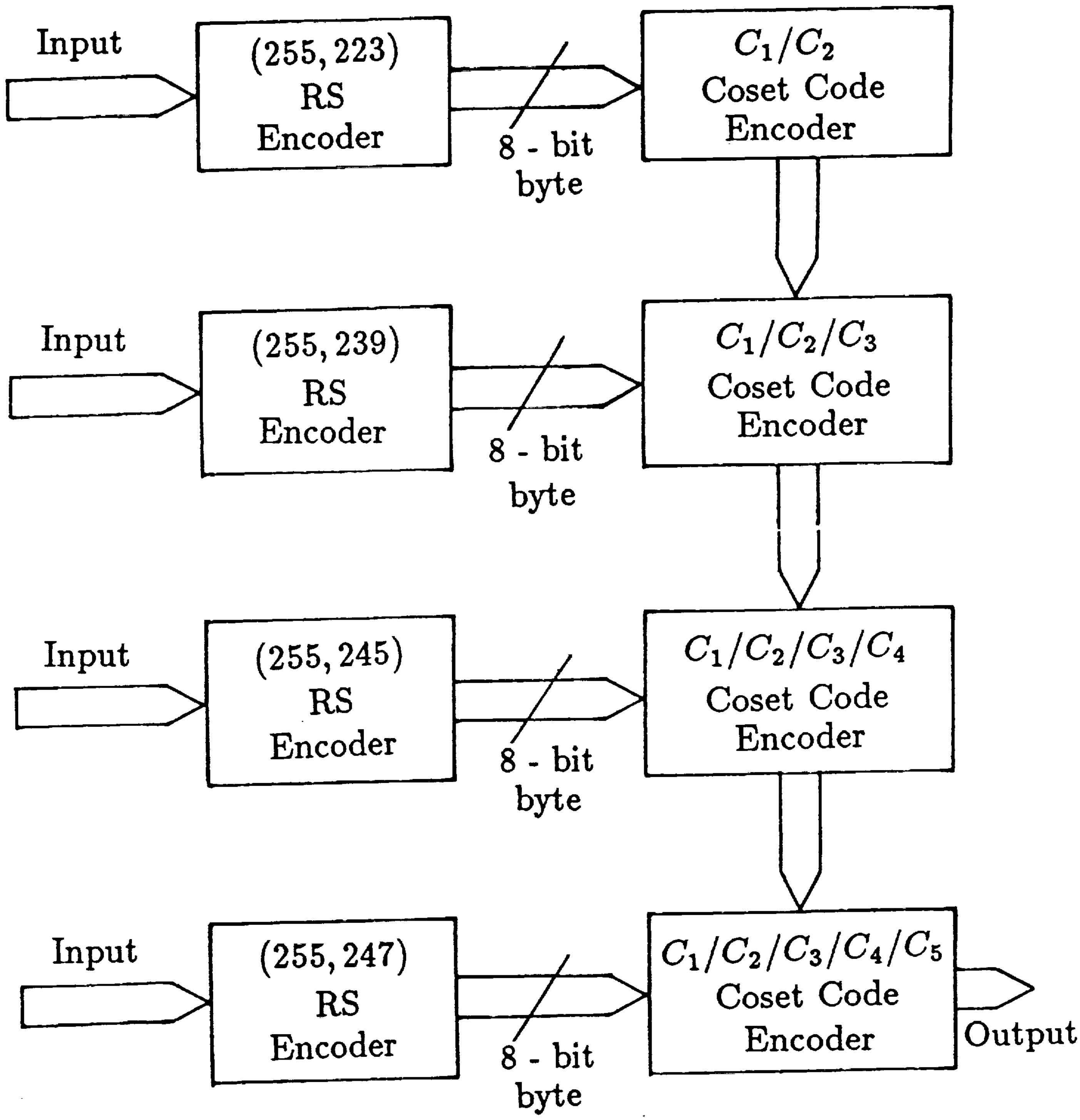


Figure 10 An encoder for the 4-level concatenated code system given in Example II

4. A Single-Level Concatenated TCM Scheme

- A multi-dimensional TCM code can be viewed as a concatenated modulation code with a convolutional code as the outer code and a block modulation code as the inner code as shown in Figure 11.

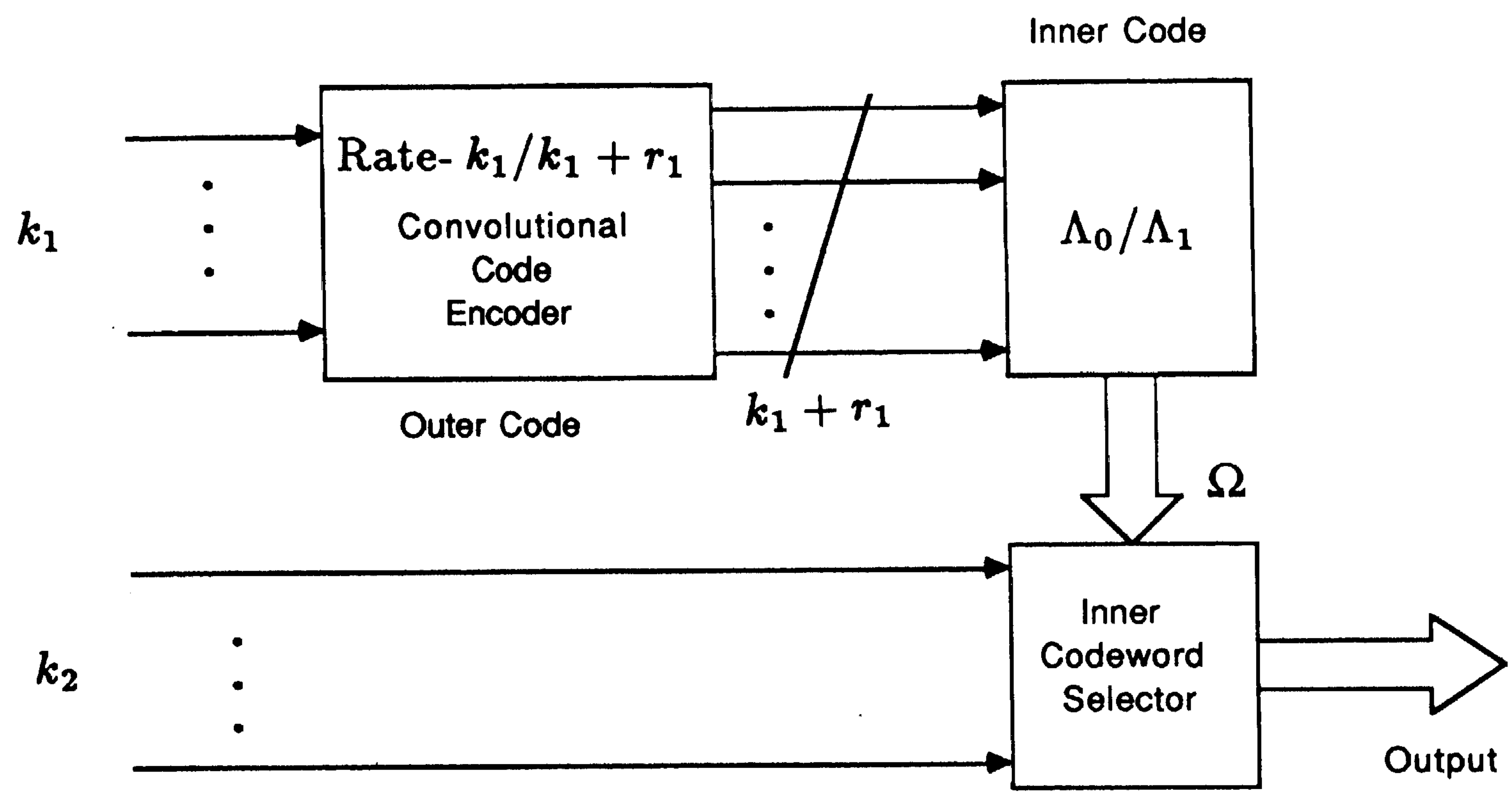


Figure 11 A single-level concatenated TCM system

- The basic components in this concatenation approach are :

  (1) A binary rate - $k_1/k_1 + r_1$ convolutional code, and

  (2) A linear block modulation code Λ_0 of length L over a certain two-dimensional elementary signal space S. The dimension of Λ_0 is

$$\dim[\Lambda_0] \triangleq \log_2 |\Lambda_0| = k_1 + k_2 + r_1.$$

- Let Λ_1 be a linear subcode of Λ_0 with

$$\dim[\Lambda_1] = \log_2 |\Lambda_1| = k_2.$$

- Let Δ_0 and Δ_1 be the minimum squared Euclidean distances of Λ_0 and Λ_1 respectively. Then $\Delta_0 \leq \Delta_1$.

- Partition Λ_0 into $2^{k_1+r_1}$ cosets modulo Λ_1. Let Λ_0/Λ_1 denote the partition. Then

$$|\Lambda_0/\Lambda_1| = 2^{k_1+r_1}$$

Encoding Operation

- During each encoding interval, a k-bit message block is applied to the input of the encoder.
- This message block is divided into two parts, a k_1-bit message sub-block and a k_2-bit message sub-block.
- First the k_1-bit message sub-block is encoded based on the rate - k_1/k_1+r_1 convolutional code into a code block of $k_1 + r_1$ bits.
- This $(k_1 + r_1)$- bit code block then selects a coset Ω from Λ_0/Λ_1 which appears at the output of the coset selector.

- At the second step of the encoding, the k_2-bit message sub-block selects a codeword from the coset Ω .
- Hence the output of the coset selector is a sequence of cosets from the partition Λ_0/Λ_1 , and the output of the codeword selector is a sequence of codewords from Λ_0 .
- All the possible code sequences at the output of the over-all encoder form a multi-dimensional TCM code with signal set Λ_0.

- A special case is $\Lambda_0 = S^L$.
- All the coset sequences at the output of the coset selector form a trellis. Each branch in the trellis corresponds to a coset in Λ_0/Λ_1.
- All the code sequences at the output of the overall encoder form the code trellis. In this code trellis, two adjacent nodes (or states) are connected by 2^{k_2} parallel branches which correspond to the 2^{k_2} codewords in a coset in Λ_0/Λ_1.

Spectral Efficiency

- Since k information bits are encoded into a sequence of L signal symbols from the two-dimensional elementary signal set S during each encoding interval, the spectral efficiency is

$$\eta = k/L$$

Minimum Free Branch Separation

- Consider a convolution code. A code sequence is simply a path in the trellis of the code.
- Define the branch separation between two paths, $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$, in the code trellis, denoted $d_B(\bar{\mathbf{u}}, \bar{\mathbf{v}})$, as the number of branches where $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ differ.
- Let $w_B(\bar{\mathbf{u}})$ denote the number of nonzero branches on the path $\bar{\mathbf{u}}$. We call $w_B(\bar{\mathbf{u}})$ the branch weight of $\bar{\mathbf{u}}$.

- The minimum free branch separation of a convolutional code C is defined as

$$d_{\text{B-free}} \triangleq \min\{d_B(\bar{\mathbf{u}}, \bar{\mathbf{v}}) : \bar{\mathbf{u}}, \bar{\mathbf{v}} \in C \text{ and } \bar{\mathbf{u}} \neq \bar{\mathbf{v}}\}$$

$$= \min\{w_B(\bar{\mathbf{v}}) : \bar{\mathbf{v}} \in C \text{ and } \bar{\mathbf{v}} \neq \bar{0}\} \tag{1}$$

Minimum Free Squared Eucliden Distance

- Let Ω and Ω' be two cosets in Λ_0/Λ_1. The squared Euclidean distance between Ω and Ω' is defined as

$$d(\Omega, \Omega') \triangleq \min\{d(\bar{\mathbf{u}}, \bar{\mathbf{v}}) : \bar{\mathbf{u}} \in \Omega \text{ and } \bar{\mathbf{v}} \in \Omega'\} \tag{2}$$

- Clearly, $d(\Omega, \Omega') \geq \Delta_0$ for $\Omega \neq \Omega'$ and $d(\Omega, \Omega') = 0$ for $\Omega = \Omega'$.

- Let

$$\bar{Z} = (\Omega_0, \Omega_1, \cdots, \Omega_i, \cdots)$$
$$\bar{Z}' = (\Omega_0', \Omega_1', \cdots, \Omega_i', \cdots)$$

be two coset sequences at the output of the coset selector.

- The squared Euclidean distance between Ω and Ω' is given by

$$d(\bar{Z}, \bar{Z}') = \sum_{i=0}^{\infty} d(\Omega_i, \Omega_i') \tag{3}$$

- Then

$$d(\bar{Z}, \bar{Z}') = \Delta_0 \cdot d_{\text{B-free}} \tag{4}$$

- Now consider the code trellis at the output of the overall encoder shown in Figure11. Since the minimum squared Euclidean distance between parallel branches is Δ_1 , the minimum squared Euclidean distance of the concatenated TCM code, denoted D_{free} , is lower bounded as

$$D_{\text{free}} \geq \min\{\Delta_1, \Delta_0 \cdot d_{\text{B-free}}\} \tag{5}$$

- The concatenation approach provides a systematic method for constructing multi-dimensional TCM codes.

- We may use multi-level block modulation codes as the inner codes. There are many effective methods for constructing multi-level modulation codes.

- Construction of convolutional codes with minimum free branch separation can be carried out in the same manner as that of convolutional codes with good free distance.

Example

- In this example, the convolutional code is a rate- $2/3$ code of constraint length $\nu = 2$ and minimum branch separation $d_{\text{B-free}} = 2$.

- The code is generated by the following transfer function matrix :

$$G(D) = \begin{pmatrix} 1+D & D & 1+D \\ D & 1 & 1 \end{pmatrix}$$

- It has a trellis diagram of 4 states.

- The block inner code Λ_0 is a basic 3-level 8-PSK code of length $L = 8$ which is formed by the following three binary component codes :

  (1) $C_{01} = (\ 8\ ,\ 4\ ,\ 4\ )$, a first-order RM code ,

  (2) $C_{02} = (\ 8\ ,\ 7\ ,\ 2\ )$, a single parity-check even weight code, and

  (3) $C_{03} = (\ 8\ ,\ 8\ ,\ 1\ )$, the vector space of all binary 8-tuples.

- Hence

$$\Lambda_0 = C_{01} * C_{02} * C_{03}$$

- This code has minimum squared Euclidean distance $\Delta_0 = 2.344$ and dimension $\dim[\Lambda_0] = 19$.

- The subcode Λ_1 of Λ_0 is the following basic 3-level 8-PSK code ,

$$\Lambda_1 = C_{11} * C_{02} * C_{03}$$

  where $C_{11} = (8,1,8)$, a repetition code.

- Note that $C_{11} \subset C_{01}$. Hence $\Lambda_1 \subset \Lambda_0$.

- Λ_1 has minimum squared Euclidean distance $\Delta_1 = 4$ and dimension $\dim[\Lambda_1] = 16$.

- Furthermore Λ_1 has a 4-state 8-section trellis diagram as shown in Figure 12 .

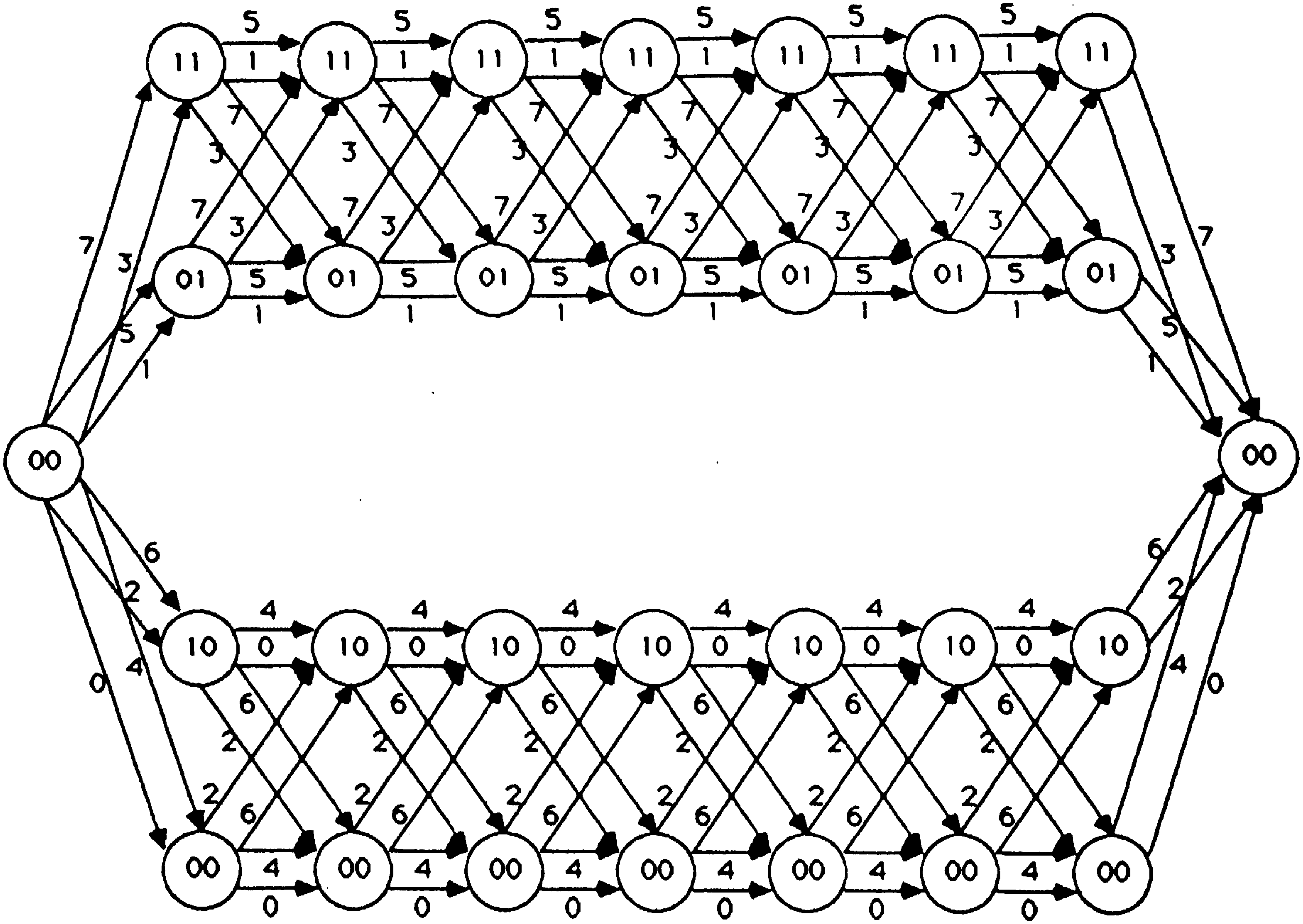


Figure 12

- Λ_0/Λ_1 consists of $2^{19}/2^{16} = 2^3 = 8$ cosets. All the cosets in Λ_0/Λ_1 have trellis diagrams isomorphic to that of Λ_1.

- Let $k_1 = 2$ and $k_2 = 16$.

- During each encoding interval, $k_1 = 2$ information bits are encoded into a $k_1 + r_1 = 3$-bit code block. The 3-bit code block selects a coset Ω from Λ_0/Λ_1. Then the other $k_2 = 16$ information bits select a codeword from Ω .

- The resultant code is a concatenated 16-dimensional 8-PSK TCM code with

$$D_{\text{free}} = \min\{4, 2 \times 2.344\} = 4$$

and spectral efficiency

$$\eta = 18/8 = 2.25 \qquad \text{bits /symbol}$$

- Both inner and outer codes can be decoded with soft-decision Viterbi algorithm.

5. Multi-Level Concatenated TCM Codes

- In a multi-level concatenated TCM scheme, we need a sequence of convolutional outer codes and a sequence of block modulation inner codes.
- For $1 \leq i \leq q$, let C_i be a rate $-k_i/n_i$ convolutional code with free branch separation $d^{(i)}_{\text{B-free}}$.
- Let Λ_0 be a linear block modulation code over an elementary signal set S with length L, dimension m_0 and minimum squared Euclidean distance Δ_0, where

$$m_0 = n_1 + n_2 + \cdots + n_q \tag{6}$$

- From Λ_0, we form a sequence of subcodes,

$$\Lambda_0, \ \Lambda_1, \ \Lambda_2, \ \cdots, \ \Lambda_q$$

- For $1 \leq i \leq q, \Lambda_i$ is a linear subcode of Λ_{i-1} with dimension

$$m_i = \dim[\Lambda_i] = m_{i-1} - n_i \tag{7}$$

and minimum squared Euclidean distance Δ_i.

- From (6) and (7), we see that

$$\begin{aligned} m_1 &= n_2 + n_3 + \cdots + n_{q-1} + n_q \\ m_2 &= n_3 + n_4 + \cdots + n_q \\ &\vdots \\ m_{q-1} &= n_q \\ m_q &= 0 \end{aligned} \tag{8}$$

- Furthermore

$$\Delta_0 \leq \Delta_1 \leq \cdots \leq \Delta_q$$

- Note that Λ_q consists of only the all-zero codewords. Hence $\Delta_q = \infty$.

- Now we form the following partitions of Λ_0 :

$$\begin{aligned}
B_1 &= \Lambda_0/\Lambda_1 \\
B_2 &= \Lambda_0/\Lambda_1/\Lambda_2 \\
&\vdots \\
B_q &= \Lambda_0/\Lambda_1/ \cdots /\Lambda_q
\end{aligned} \tag{9}$$

- For $1 \le i \le q$,

$$B_i = \Lambda_0/\Lambda_1/ \cdots /\Lambda_i$$

is called a coset code which consists of

$$2^{m_0 - m_i} = 2^{n_1 + n_2 + \cdots + n_i} \tag{9}$$

- The intra-set distance of each coset in B_i is Δ_i (the minimum squared Euclidean distance of Λ_i.

- The minimum squared separation between the cosets in B_i is

$$d_s(B_i) = \min\{d(\Omega, \Omega') : \Omega, \Omega' \in B_i \text{ and } \Omega \ne \Omega'\} \tag{10}$$

Multi-Level Concatenation

- Now, we use the coset codes,

$$B_1,\ B_2,\ \cdots\ B_q$$

  as inner codes in the multi-level concatenation.

- At the i-th level of concatenation, the convolutional code C_i is used as the outer code and B_i is used as the inner code.

- The output of the i-th level encoder is an input to the $(i+1)$-th level encoder.

- An organization of the q-level concatenated TCM system is shown in Figure 13.
- Every inner code encoder (or coset selector), except the first level, has two inputs, one from the output of an outer code encoder and one from the output of the inner code encoder of the preceding level.

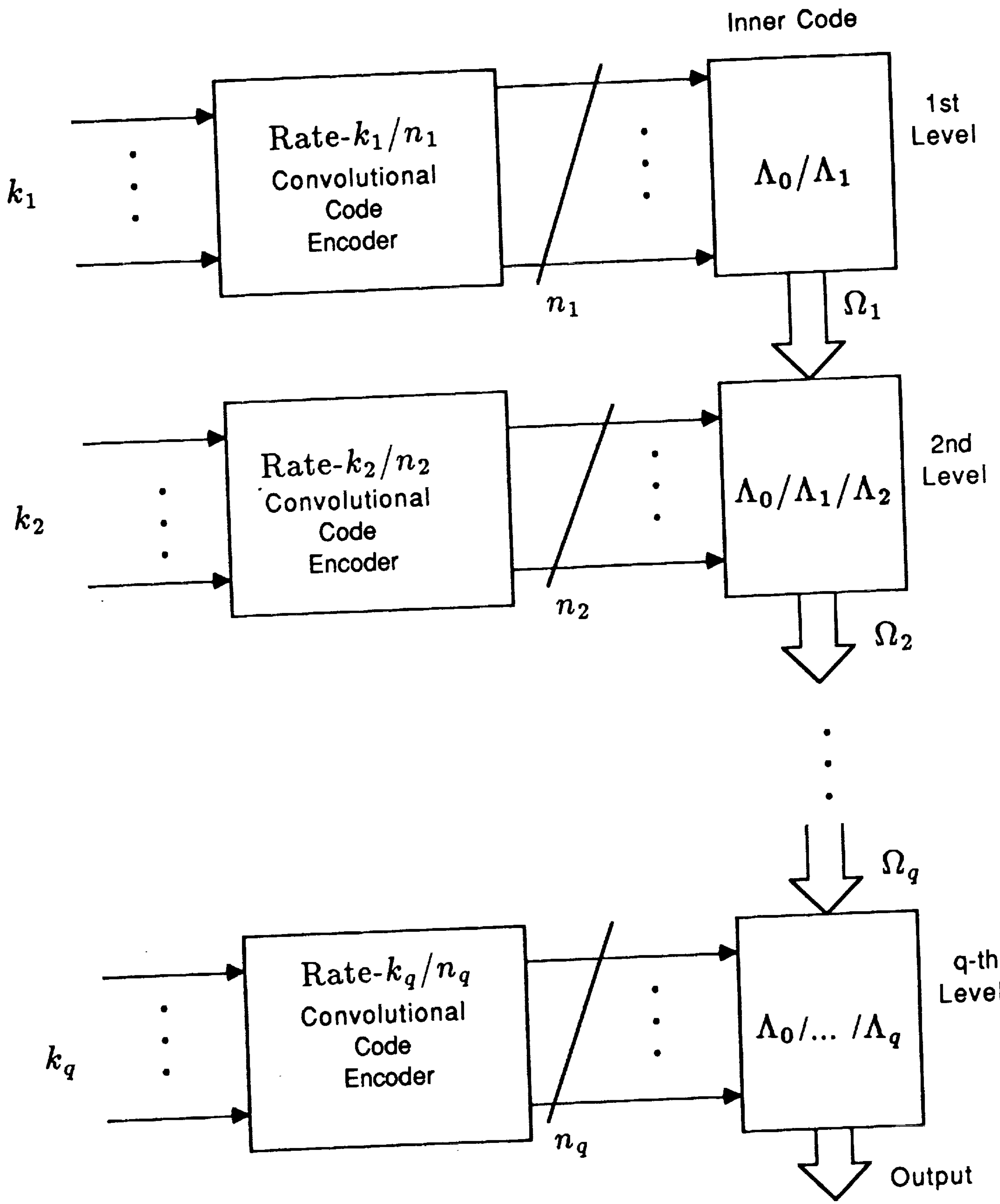


Figure 13 A multi-level-level concatenated TCM system

Encoding Operation

- For $1 \leq i \leq q$, the i-th level encoding is accomplished in two steps :

  (1) At the time l, a message of k_i bits is encoded based on the convolutional outer code C_i into an n_i-bit code block.

  (2) The n_i-bit code block selects a coset from the coset code

$$B_i = \Lambda_0/\Lambda_1/ \cdots /\Lambda_i$$

- The output of the i-th level inner code encoder is a sequence of cosets from B_i. All the possible coset seuences form a trellis, each branch in the trellis corresponds to a coset in B_i.

- The output of the q-th level inner code encoder (the output of the overall encoder) is a sequence of codewords from $B_q = \Lambda_0$. All the possible output sequences at the q-th level form a multi-dimensional TCM code with a trellis isomorphic to that of the convolutional outer code C_q.

- The minimum free squared Euclidean distance of the coset trellis code at the i-th level is

$$D_{\text{free}}^{(i)} \geq \Delta_{i-1} \cdot d_{\text{B-free}}^{(i)} \tag{11}$$

- The minimum squared Euclidean distance of the overall TCM code is

$$D_{\text{free}} = \min_{1 \leq i \leq q} \{\Delta_{i-1} \cdot d_{\text{B-free}}^{(i)}\} \tag{12}$$

6. Decoding of Multi-Level Concatenated TCM Codes

- Consider a multi-level concatenated code. Let

$$\bar{\mathbf{V}} = (\bar{\mathbf{v}}_0, \bar{\mathbf{v}}_1, \cdots, \bar{\mathbf{v}}_\ell, \cdots)$$

  be the transmitted code sequence where $\bar{\mathbf{v}}_\ell$ is a codeword in one of the cosets of the coset code $B_1 = \Lambda_0/\Lambda_1$.

- Let

$$\bar{\mathbf{R}} = (\bar{\mathbf{r}}_0, \bar{\mathbf{r}}_1, \cdots, \bar{\mathbf{r}}_\ell, \cdots)$$

  be the received sequence.

- Decoding is carried out in q steps, from the first-level to the q-th level.

First-Level Decoding

- Decode $\bar{\mathbf{r}}_l$ into one of the cosets in $B_1 = \Lambda_0/\Lambda_1$.

- Based on the decoded coset, we identify the output code block $\bar{\mathbf{a}}_l^{(1)}$ of the convolutional outer code encoder C_1(inverse mapping).

- Decode the sequence

$$(\bar{\mathbf{a}}_0^{(1)}, \bar{\mathbf{a}}_1^{(1)}, \cdots, \bar{\mathbf{a}}_l^{(1)}, \cdots)$$

  based on the convolutional outer code C_1.

- Let

$$(\bar{\mathbf{b}}_0^{(1)}, \bar{\mathbf{b}}_1^{(1)}, \cdots, \bar{\mathbf{b}}_l^{(1)}, \cdots)$$

  be the decoded sequence.

- The input information sequence can be retrieved from this decoded sequence.
- Furthermore, the decoded sequence $(\bar{\mathbf{b}}_0^{(1)}, \bar{\mathbf{b}}_1^{(1)}, \dots, \bar{\mathbf{b}}_l^{(1)}, \dots)$ reproduces a coset sequence,

$$(\Omega_0^{(1)}, \Omega_1^{(1)}, \dots, \Omega_l^{(1)}, \dots),$$

  at the output of the first-level decoder, where $\Omega_l^{(1)} \in \Lambda_0/\Lambda_1$.
- This coset sequence is then applied at the input of the second-level decoder.

Second-Level Decoding

- Based on input information $\Omega_l^{(1)}$, decode $\bar{\mathbf{r}}_l$ into one of cosets in $\Omega_l^{(1)}/\Lambda_2$.

- Based on the decoded coset, we identify the output code block $\bar{\mathbf{a}}_l^{(2)}$ of the convolutional outer code encoder C_2 (inverse mapping).

- Decode the sequence

$$(\bar{\mathbf{a}}_0^{(2)}, \bar{\mathbf{a}}_1^{(2)}, \cdots, \bar{\mathbf{a}}_l^{(2)}, \cdots)$$

  based on the convolutional outer code C_2.

- Let

$$(\bar{\mathbf{b}}_0^{(2)}, \bar{\mathbf{b}}_1^{(2)}, \cdots, \bar{\mathbf{b}}_l^{(2)}, \cdots)$$

  be the decoded sequence.

- Retrieve the second-level input information sequence from this decoded sequence.

- Based on $(\bar{\mathbf{b}}_0^{(2)}, \bar{\mathbf{b}}_1^{(2)}, \cdots, \bar{\mathbf{b}}_l^{(2)}, \cdots)$, we reproduces a coset sequence,

$$(\Omega_0^{(2)}, \Omega_1^{(2)}, \cdots, \Omega_l^{(2)}, \cdots),$$

  at the output of the second-level decoder, where $\Omega_l^{(2)} \in \Lambda_0/\Lambda_1/\Lambda_2$.

- This coset sequence is then applied at the input of the third-level decoder.

- Other levels of decoding are carried out in the same manner.

Remarks

(1) Decoded information is passed from one level to another level.

(2) Decoding at each level depends on the decoded information from the preceding level.

(3) Error propagation may occur.

(4) To reduce the probability of error propagation, the first few level outer codes should be powerful.